Analysis of Optical Waveguide Consisting of a Square-Law Lenslike Medium and Its Analogies to Circular TE_{01} Waveguide

SHINNOSUKE SAWA, MEMBER, IEEE, AND NOBUAKI KUMAGAI, SENIOR MEMBER, IEEE

Abstract—**Propagation behavior of light beams along sinusoidal and serpentine bends as well as circular bends and linearly tapered bends of optical waveguides consisting of a square-law lenslike medium is investigated in detail, both theoretically and numerically, on the basis of the approximate wave theory. A new design method of the circular bend for removing the effects of the bend is proposed and numerical results are presented. The divergence phenomena of the beam trajectory in both the sinusoidal and serpentine bends of the optical waveguide are discussed in comparison with mode-conversion phenomena occurring in the circular TE_{01} waveguide with the same bends. Several design conditions to eliminate undulations of the beam trajectory and/or the spot size which would occur at a circular bend of the optical waveguide are also studied, and interesting analogies to the design conditions proposed so far to prevent mode-conversion losses at a circular bend of the TE_{01} waveguide are shown.**

I. INTRODUCTION

IT IS EXPECTED that dielectric waveguides operating at optical frequencies will in the future constitute one of the major transmission systems. In order to transmit a light wave along a dielectric, it is necessary to achieve a suitable variation of the permittivity (refractive index) in the transverse cross section of the dielectric material [1]. The permittivity need not vary stepwise, but may decrease continuously in inverse proportion to the square of the distance from the center axis of the medium. A medium with such a permittivity profile is equivalent to an ordinary optical lens and hence is termed a (square-law) lenslike medium. A typical example of a lenslike medium achieved with a solid is SELFOC, which was developed jointly by the Nippon Electric Co. and the Nippon Plate Glass Co. [2], and one achieved with gas is the gas-lens beam waveguide developed by Bell Telephone Laboratories [3].

Two analytical approaches are possible to clarify the propagation behavior of light beams along lenslike media. One is the geometrical-optics approach [4]–[6] and the other is the wave-optics approach [7]–[10]. In the first approach, the light beam to be transmitted is treated as an optical ray, while in the second approach it is treated as an electromagnetic wave. The geometrical-optics approach is sufficient to clarify only the behavior of the trajectory of the beam center (the so-called beam trajectory). However, the wave-optics approach is necessary in order to clarify the modal behavior of the light beam such as mode conversion and the propagation constant as well as the response of the electromagnetic fields.

Manuscript received September 10, 1975; revised December 16, 1975.
S. Sawa is with the Department of Electronics, Faculty of Engineering, Ehime University, Matsuyama City, 790 Japan.
N. Kumagai is with the Department of Electrical Communication, Faculty of Engineering, Osaka University, Suita City, 565 Japan.

In the present paper, we investigate in detail the propagation behavior of light beams along bends of square-law lenslike media from the viewpoint of wave theory. General expressions for the responses of electromagnetic fields of light beams along curved lenslike media are derived, following the approximate wave theory previously described [8], [10]–[12]. The results are applied to a sinusoidal bend, a serpentine bend due to the weight of the guiding system itself, a circular bend and a linearly tapered bend, and the propagation behavior of light beams is studied in detail theoretically and numerically, compared with the results obtained so far from the viewpoint of ray theory [4]–[6]. A new design method of the circular bend for removing the effects of the bend is proposed, which makes it possible to connect the circularly bent section to the straight section without off-setting and tilting the center axis of the bend, unlike the previous methods [5], [9]. Numerical results for this design method are also presented. Further, the divergence phenomena of the beam trajectory occurring in the sinusoidal bend and the serpentine bend [5], [6] and the design conditions for the circular bend of the optical waveguide with a square-law lenslike medium [5], [9], [12], [20] are discussed in comparison with the millimeter-wave transmission system using circular TE_{01} waveguides [13]–[19], and as a result various analogies between the two guiding systems are shown.

For simplicity, two-dimensional lenslike media are used and the analysis is limited to the paraxial beam approximations throughout the paper.

II. GENERAL EXPRESSIONS FOR THE RESPONSES OF ELECTROMAGNETIC FIELDS OF LIGHT BEAMS ALONG CURVED LENSLIKE MEDIA

Let us consider a two-dimensional model of the bend section of a square-law lenslike medium as shown in Fig. 1. We assume that the radius of curvature of the bend varies slowly as a function of z. Let the permittivity of the medium be expressed as

$$\varepsilon = \varepsilon_c[1 - g^2(r - R)^2] \quad (1)$$

where ε_c represents the constant permittivity on the center axis of the medium $r = R$ (on-axis permittivity), and g is a focusing parameter specifying the rate of change of permittivity in the transverse x direction. R denotes the radius

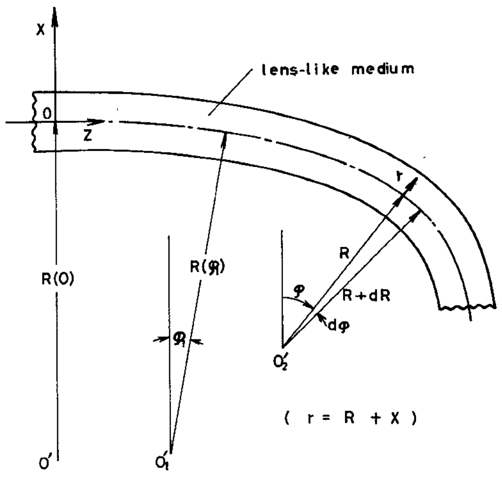


Fig. 1. Curved section of the optical waveguide consisting of a square-law lenslike medium.

of curvature of the bend, being a function of z or ϕ and represented as $R(z)$ or $R(\phi)$.

We also assume that the variations of the permittivity ε in the r and ϕ (or z) directions are small enough to be neglected over a distance of a free-space wavelength λ_0 of the light beam. Then, the scalar wave equation which determines the responses of the electromagnetic field of the light beam is expressed approximately in polar coordinates (r,ϕ) as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\omega^2\mu\varepsilon_c[1-g^2(r-R)^2]V=0 \tag{2}$$

where sinusoidal time dependence of the fields with angular frequency ω is assumed, and μ denotes the permeability of the medium.

By performing the transformation of variables from (r,ϕ) to (ξ,z) as

$$\xi = R(\phi)\ln\frac{r}{R(\phi)} \tag{3}$$

$$z=\int_0^{\phi} R(\phi_0)\,d\phi_0 \tag{4}$$

we can rewrite (2) as

$$\begin{aligned}\frac{\partial^2 V}{\partial \xi^2}&+\frac{\partial^2 V}{\partial z^2}+k^2(0)\exp\left[\frac{2\xi}{R(z)}\right]\\&\cdot\left[1-g^2R^2(z)\left(\exp\left[\frac{\xi}{R(z)}\right]-1\right)^2\right]V\\&+\frac{R'(z)}{R(z)}\left\{2(\xi-R(z))\frac{\partial^2 V}{\partial\xi\,\partial z}+\frac{\partial V}{\partial z}\right\}\\&+\left\{\frac{R'(z)}{R(z)}\right\}^2(\xi-R(z))^2\frac{\partial^2 V}{\partial \xi^2}\\&+\left[\left\{\frac{R'^2(z)}{R^2(z)}+\frac{R''(z)}{R(z)}\right\}(\xi-R(z))-\frac{R'^2(z)}{R(z)}\right]\frac{\partial V}{\partial \xi}=0\end{aligned} \tag{5}$$

with

$$k(0)=\omega\sqrt{\mu\varepsilon_c} \tag{6}$$

where the primes indicate the differentiation with respect to z.

Let us expand the exponential terms $\exp[2\xi/R(z)]$ and $\exp[\xi/R(z)]-1$ in power series of $\xi/R(z)$ and omit the terms higher than third order by noticing that $\xi/R(z)\ll 1$ when $x/R(z)\ll 1$ $(x=r-R)$ since

$$\begin{aligned}\xi&=R(z)\ln\left(1+\frac{x}{R(z)}\right)\\&=x-\frac{x^2}{2R(z)}+\frac{x^3}{3R^2(z)}-\frac{x^4}{4R^3(z)}+\cdots.\end{aligned} \tag{7}$$

Further, we discard the terms with $R'(z)$, $R'^2(z)$, and $R''(z)$ in (5), by assuming

$$\begin{aligned}&\sqrt{g\lambda_0}\cdot|R'(z)|\ll 1 \qquad \sqrt{g\lambda_0}\cdot|R''(z)|\ll 1\\&\lambda_0\cdot\left|\frac{R'(z)}{R(z)}\right|\ll 1 \qquad |R'(z)|\ll 1.\end{aligned} \tag{8}$$

As a result, we have the simplified wave equation as

$$\begin{aligned}\frac{\partial^2 V}{\partial \xi^2}&+\frac{\partial^2 V}{\partial z^2}+k^2(0)\\&\cdot\left[1+\frac{2}{R(z)}\xi-g^2\left\{1-\frac{2}{g^2R^2(z)}\right\}\xi^2\right]V=0.\end{aligned} \tag{9}$$

If we put

$$V(\xi,z)=\tilde{U}(\xi,z)\exp[-jk(0)z](\equiv U(x,z)\exp[-jk(0)z]) \tag{10}$$

with the assumption

$$\left|\frac{\partial^2\tilde{U}}{\partial z^2}\right|\ll 2k(0)\left|\frac{\partial\tilde{U}}{\partial z}\right| \quad \left(\text{or } \left|\frac{\partial^2 U}{\partial z^2}\right|\ll 2k(0)\left|\frac{\partial U}{\partial z}\right|\right) \tag{11}$$

and substitute (9) into (8), we see that the field-distribution function $\tilde{U}(\xi,z)$ must satisfy the paraxial wave equation

$$\begin{aligned}\frac{\partial^2\tilde{U}}{\partial \xi^2}&-j2k(0)\frac{\partial\tilde{U}}{\partial z}-k^2(0)\\&\cdot\left[-\frac{2}{R(z)}\xi+g^2\left\{1-\frac{2}{g^2R^2(z)}\right\}\xi^2\right]\tilde{U}=0.\end{aligned} \tag{12}$$

For convenience, let us express the field-distribution function $\tilde{U}(\xi,0)$ of the input beam at $z=0$ as

$$\begin{aligned}\tilde{U}(\xi,0)&=\exp\left[-\frac{\{\xi-\Delta(0)\}^2}{2\sigma^2(0)}-jk(0)\Delta'(0)\xi\right]\\&\cdot \mathrm{He}_\nu\left[\frac{\xi-\Delta(0)}{\sigma(0)}\right]\end{aligned} \tag{13}$$

where $\sigma(0)$, $\Delta(0)$, and $\Delta'(0)$ are constants independent of ξ and z, and $\mathrm{He}_\nu(X)$ refers to the Hermite polynomial of the νth order, defined as in [11, eq. (A2)].

Following the convenient method of analysis based on approximate wave theory [10]–[12] and with the help of the Wentzel–Kramers–Brillouin–Jeffreys (WKBJ) method [21], we derive the field-distribution function $\tilde{U}(\xi,z)$, from which the field distribution function $U(x,z)$ can be obtained by approximating the variable ξ by the first two terms of its expansion (7) as $\xi \cong x - x^2/[2R(z)]$. The result is given below.

$$U(x,z) = \rho^{1/4}(z)\cdot \frac{\left(\cos g_0\theta + u(0)\sin g_0\theta + j\dfrac{w_c^2}{\sigma^2(0)}\sin g_0\theta\right)^{\nu/2}}{\left(\cos g_0\theta + u(0)\sin g_0\theta - j\dfrac{w_c^2}{\sigma^2(0)}\sin g_0\theta\right)^{(\nu+1)/2}}$$
$$\cdot \exp\left[-\frac{\{x-\delta(z)\}^2}{2s^2(z)} - jk(0)\hat{\delta}'(z)\left\{1-\frac{\delta(z)}{R(z)}\right\}^2 x - \frac{\delta^3(z)/R(z)}{2\sigma^2(z)\{1-\delta(z)/R(z)\}^2} + j\frac{k(0)}{2}\right.$$
$$\left.\cdot\left[\delta(z)\hat{\delta}'(z)\left\{1-\frac{\delta(z)}{R(z)}\left(1-\frac{\delta(z)}{R(z)}\right)\right\} - \delta(0)\hat{\delta}'(0) - \psi(z)\right]\right]$$
$$\cdot \mathrm{He}_\nu\left[\frac{\sqrt{\rho(z)}\left\{x - \delta(z)/(1-\delta(z)/R(z)) - \dfrac{x^2}{2R(z)}\right\}}{\sigma(0)\sqrt{(\cos g_0\theta + u(0)\sin g_0\theta)^2 + \dfrac{w_c^4}{\sigma^4(0)}\sin^2 g_0\theta}}\right] \tag{14}$$

where

$$\frac{1}{s^2(z)} = \frac{1}{\sigma^2(z)}\left\{1-\frac{\delta(z)}{R(z)}\right\}^{-1} - jk(0)\cdot\frac{\hat{\delta}'(z)}{R(z)}\left\{1-\frac{\delta(z)}{R(z)}\right\} \tag{15}$$

$$\delta(z) = \Delta(z)\left\{1+\frac{\Delta(z)}{R(z)}\right\}^{-1} \tag{16}$$

with

$$\frac{1}{\sigma^2(z)} = \frac{\rho(z)}{\sigma^2(0)}\,\frac{\left[\cos g_0\theta - u(z)\sin g_0\theta - j\dfrac{\sigma^2(0)}{w_c^2}\{(1+u(0)u(z))\sin g_0\theta + (u(z)-u(0))\cos g_0\theta\}\right]}{\cos g_0\theta + u(0)\sin g_0\theta - j\dfrac{w_c^2}{\sigma^2(0)}\sin g_0\theta} \tag{17}$$

$$\Delta(z) = \rho^{-1/2}(z)\left[\left(\cos g_0\theta + \frac{\rho'(0)}{2g_0}\sin g_0\theta\right)\cdot\delta(0)\left\{1-\frac{\delta(0)}{R(0)}\right\}^{-1} + \frac{1}{g_0}\cdot\left[\hat{\delta}'(0)\left\{1-\frac{\delta(0)}{R(0)}\right\} - f'(0)\right]\sin g_0\theta\right] + f(z) \tag{18}$$

$$\frac{1}{\sigma^2(0)} = \frac{1}{s^2(0)}\left\{1-\frac{\delta(0)}{R(0)}\right\} + jk(0)\cdot\frac{\hat{\delta}'(0)}{R(0)}\left\{1-\frac{\delta(0)}{R(0)}\right\} \tag{19}$$

where

$$\hat{\delta}'(z) = [\delta'(z) - R'(z)\delta^2(z)/R^2(z)]\{1-\delta(z)/R(z)\}^{-3}$$

$$f(z) = \frac{1}{g_0\sqrt{\rho(z)}}\int_0^z \frac{\sin[g_0\{\theta(z)-\theta(\eta)\}]}{R(\eta)\sqrt{\rho(\eta)}}\,d\eta,$$

$$f'(0) = \left.\frac{df(z)}{dz}\right|_{z=0}, \qquad \rho(z) = \frac{g}{g_0}\sqrt{1-\frac{2}{g^2R^2(z)}},$$

$$\rho'(0) = \left.\frac{d\rho(z)}{dz}\right|_{z=0}, \qquad \delta'(0) = \left.\frac{d\delta(z)}{dz}\right|_{z=0}, \qquad u(z) = \frac{\rho'(z)}{2g_0\rho^2(z)},$$

$$\theta(z) \equiv \theta = \int_0^z \rho(\eta)\,d\eta, \qquad \psi(z) = \int_0^z \frac{\delta(\eta)}{R(\eta)}\,d\eta,$$

$$g_0 = g\sqrt{1-\frac{2}{g^2R^2(0)}}, \qquad w_c = \frac{1}{\sqrt{g_0k(0)}}. \tag{20}$$

If we restrict our attention to a mild bend whose curvature is small enough to satisfy $\delta(z)/R(z) \ll 1$ and $\delta'(z)/[g_0R(z)] \ll 1$ together with $x/R(z) \ll 1$, we can adequately approximate $1/s^2(z)$ and $\delta(z)$ in the previous expressions as

$$\frac{1}{s^2(z)} \cong \frac{\rho(z)}{s^2(0)}\cdot\frac{\left[\cos g_0\theta - u(z)\sin g_0\theta - j\dfrac{s^2(0)}{w_c^2}\{(1+u(z)u(0))\sin g_0\theta + (u(z)-u(0))\cos g_0\theta\}\right]}{\cos g_0\theta + u(0)\sin g_0\theta - j\dfrac{w_c^2}{s^2(0)}\sin g_0\theta} \tag{21}$$

$$\delta(z) \cong \rho^{-1/2}(z)\left[\left(\cos g_0\theta + \frac{\rho'(0)}{2g_0}\sin g_0\theta\right)\delta(0) + \left(\frac{\delta'(0)-f'(0)}{g_0}\right)\sin g_0\theta\right] + f(z). \tag{22}$$

For this case, (14) may be regarded approximately as a Hermite–Gaussian field distribution in the transverse x direction, representing the response of the light beam whose input condition is given by

$$U(x,0) = \exp\left[-\frac{\{x-\delta(0)\}^2}{2s^2(0)} - jk(0)\delta'(0)x\right] \cdot \mathrm{He}_\nu\left[\frac{x-\delta(0)}{s(0)}\right] \quad (23)$$

where $1/s^2(0)$ is an input wavefront coefficient [11], and $\delta'(0)$ and $\delta(0)$ are the input slope and input displacement of the beam center from the optic axis $x = 0$.

III. Propagation Behavior of Light Beams Along Curved Lenslike Media

A. Sinusoidal Bend

Let us consider a sinusoidal bend as shown in Fig. 2, in which the curvature $1/R(z)$ varies sinusoidally with z as

$$\frac{1}{R(z)} = \frac{1}{R_m}\sin\left(\frac{2\pi z}{p}\right) \quad (24)$$

where $1/R_m$ and p are constants, denoting the maximum value of curvature and the period of the bend, respectively. For simplicity, it is assumed that

$$\frac{1}{g^2R_m^2}\sin^2\left(\frac{2\pi z}{p}\right) \ll 1. \quad (25)$$

Substituting (24) into (14)–(22) and taking the previous assumption into consideration, we can derive the field-distribution function $U(x,z)$ of the light beam with the input condition of (23) as

$$U(x,z) = \frac{\left(\cos gz + j\frac{w^2}{s^2(0)}\sin gz\right)^{\nu/2}}{\left(\cos gz - j\frac{w^2}{s^2(0)}\sin gz\right)^{(\nu+1)/2}} \cdot \exp\left[-\frac{\{x-\delta(z)\}^2}{2s^2(z)} - jk(0)\delta'(z)x + j\frac{k(0)}{2}\{\delta(z)\delta'(z) - \delta(0)\delta'(0)\}\right] \cdot \mathrm{He}_\nu\left[\frac{x-\delta(z)}{s(0)\sqrt{\cos^2 gz + \frac{w^4}{s^4(0)}\sin^2 gz}}\right] \quad (26)$$

where

$$\frac{1}{s^2(z)} = \frac{1}{s^2(0)}\left(\frac{\cos gz - j\frac{s^2(0)}{w^2}\sin gz}{\cos gz - j\frac{w^2}{s^2(0)}\sin gz}\right) \quad (27)$$

$$\delta(z) = \delta(0)\cos gz + \frac{\delta'(0)}{g}\sin gz + \tilde{\delta}(z) \quad (28)$$

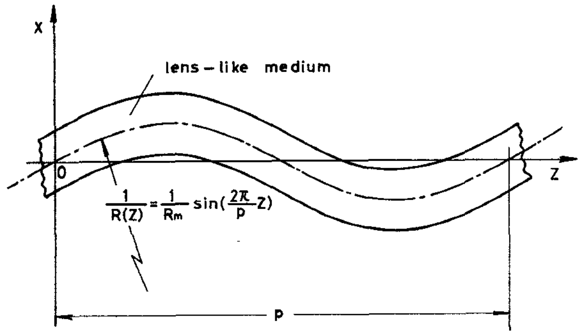


Fig. 2. Sinusoidal bend of the optical waveguide consisting of a lenslike medium.

$$\tilde{\delta}(z) = \begin{cases} \dfrac{g\sin\frac{2\pi z}{p} - \frac{2\pi}{p}\sin gz}{gR_m(g^2 - 4\pi^2/p^2)}, & \text{for } p \neq \dfrac{2\pi}{g} \quad (29) \\ \dfrac{\sin gz - gz\cos gz}{2g^2R_m}, & \text{for } p = \dfrac{2\pi}{g} \quad (30) \end{cases}$$

$$\delta'(z) = \frac{d}{dz}\delta(z) \quad (31)$$

with

$$w = 1/\sqrt{gk(0)}. \quad (32)$$

From (27) the spot size of a Gaussian beam is calculated as

$$w(z) = \frac{1}{\mathrm{Re}^{1/2}\{1/s^2(0)\}} \cdot \left[\frac{1}{2}\left\{1 + \frac{w^4}{|s(0)|^4} + \left(1 - \frac{w^4}{|s(0)|^4}\right)\cos 2gz\right\} + \mathrm{Im}\left\{\frac{w^2}{s^2(0)}\right\}\sin 2gz\right]^{1/2}. \quad (33)$$

The trajectory of the beam center (the beam trajectory) is given by (28)–(30). By setting $w(0) = w$, $\delta(0) = 0$, $\delta'(0) = 0$, and $R_m = \infty$ in the above results, we get the propagation constant β_ν for the normal modes in the straight section as

$$\beta_\nu = k(0) - g(\nu + \tfrac{1}{2}) \quad (34)$$

with $\nu = 0,1,2,\cdots$.

From (26)–(33), the following conclusions are derived, regardless of the input conditions of the light beam.

1) When the period of the bend p is not equal to $2\pi/g$, the beam trajectory $\delta(z)$ undulates about the center axis of the medium; the light beam does not diverge from the center axis and hence stable transmission can be realized.

2) When the bending period p is just equal to $2\pi/g$, the beam trajectory deviates from the center axis of the medium, undulating increasingly with the transmission distance; in other words, the so-called divergence phenomenon of the beam trajectory arises. As a result, the diffraction loss due to the finiteness of the cross section of the medium for

practical use increases without limit, and hence stable and low-loss transmission becomes impossible.

These conclusions are in complete agreement with those previously obtained by the ray-theory approach [5].

B. Serpentine Bend

Next we consider a serpentine bend as shown in Fig. 3. Such a bend would occur inevitably in the practical guiding system supported or suspended with equal spacing L. The center axis of the guiding system is bent along an elastic curve caused by its own weight, which is given by the theory of elasticity [14] as

$$x_0(z) = -K\left(\frac{z}{L}\right)^2\left(1-\frac{z}{L}\right)^2 \tag{35}$$

with

$$K = \frac{WL^4}{24eI} \tag{36}$$

where W, e, and I are, respectively, the weight per unit length, the modulus of elasticity, and the moment of inertia of the guiding system.

The curvature of the elastic curve (35) is computed as

$$\frac{1}{R(z)} \cong -\frac{12K}{L^2}\left\{1-6\left(\frac{z}{L}\right)\left(1-\frac{z}{L}\right)\right\} \tag{37}$$

where we have assumed

$$\left|\frac{dx_0(z)}{dz}\right|^2 \ll 1 \qquad \left(\frac{K^2}{L^2} \ll 1\right). \tag{38}$$

In the interval $0 \leq z \leq L$, (37) may be expanded in the Fourier series

$$\frac{1}{R(z)} = -\frac{12K}{L^2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2\pi n}{L}z\right). \tag{39}$$

Substituting (39) into (14)–(22), we have the field-distribution function $U(x,z)$ for this case. The result is expressed by (26), in which $\delta(z)$ is replaced with

$$\delta(z) = \{\delta(0)-\tilde{\delta}(0)\}\cos gz + \frac{\delta'(0)}{g}\sin gz + \tilde{\delta}(z) \tag{40}$$

where

$$\tilde{\delta}(z) = \begin{cases} \dfrac{12K}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{\cos\left(\frac{2\pi n}{L}z\right)}{n^2(4\pi^2n^2-g^2L^2)}, & \text{for } L \neq \dfrac{2\pi n}{g} \quad (41a) \\[2ex] \dfrac{12K}{\pi^2}\displaystyle\sum_{n\neq n'}^{\infty}\frac{\cos\left(\frac{2\pi n}{L}z\right)}{n^2(4\pi^2n^2-g^2L^2)} - \frac{gz\sin gz}{2(gLn')^2}, & \text{for } L = \dfrac{2\pi n'}{g}. \quad (41b) \end{cases}$$

Equation (40) represents the trajectory of the beam center. The spot size is given by the same equation as (33).

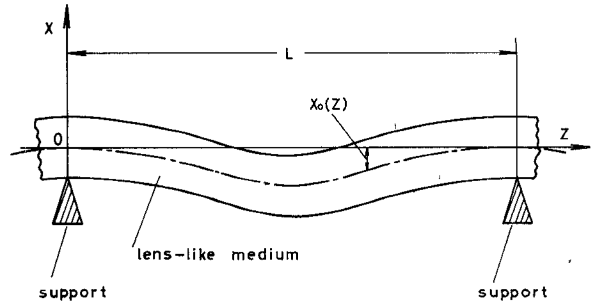


Fig. 3. Serpentine bend of the optical waveguide consisting of a lenslike medium.

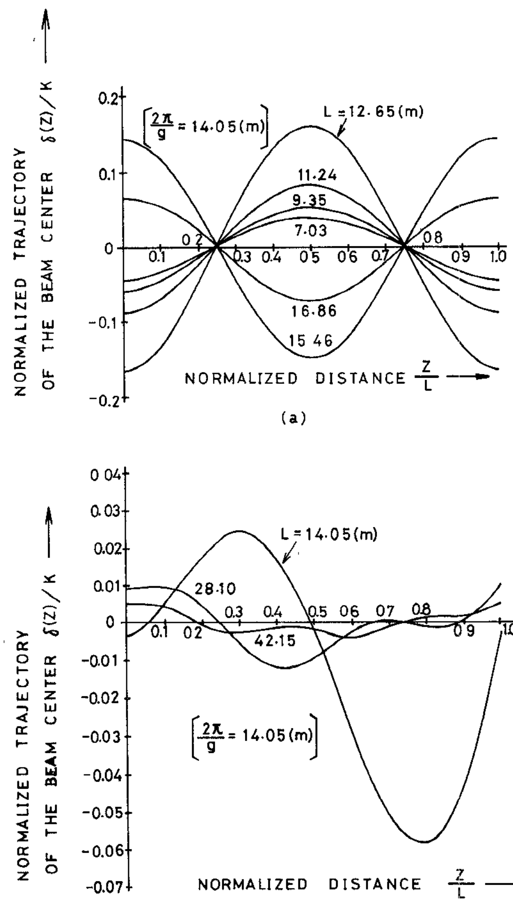


Fig. 4. Normalized beam trajectories of the light beam along the serpentine bend of the optical waveguide consisting of a lenslike medium. Input conditions of the light beam are assumed to be $\delta(0) = \tilde{\delta}(0)$ and $\delta'(0) = 0$. (a) For the case of $L \neq 2\pi n/g$. (b) For the case of $L = 2\pi n/g$.

In particular, for a light beam satisfying the input conditions

$$\delta(0) = \tilde{\delta}(0) \qquad \delta'(0) = 0 \tag{42}$$

we have

$$\delta(z) = \tilde{\delta}(z). \tag{43}$$

Therefore, for the case of $L \neq 2\pi n/g$, the beam trajectory repeats over the support or suspension interval as shown in Fig. 4(a), while for the case of $L = 2\pi n/g$ the trajectory deviates from the center axis of the medium, increasing in amplitude of undulation as the light beam propagates, as shown in Fig. 4(b).

Generally, from (40), (41a), and (41b), we obtain the following conclusions regardless of the input conditions.

1) When the support interval L is not equal to an integral multiple of $2\pi/g$ $[L \neq 2\pi n/g;\ n = 1,2,3,\cdots]$, the beam trajectory undulates around the center axis of the medium; in other words, the light beam does not diverge from the center axis and hence stable transmission is obtained.

2) When L is just equal to an integral multiple of $2\pi/g$ $[L = 2\pi n/g;\ n = 1,2,3,\cdots]$, the divergence phenomenon of the beam trajectory occurs; in other words, the light beam fluctuates increasingly with the transmission distance, deviating further from the center axis of the guiding system. As a result, for this case, too, the diffraction loss increases indefinitely and hence low-loss transmission cannot be possible. Here, it must be noted that in the case of the sinusoidal bend the divergence phenomenon of the beam trajectory occurs only when the bending period p is equal to $2\pi/g$, as analyzed already; whereas, in the case of the serpentine bend this phenomenon occurs not only when the support interval L is equal to $2\pi/g$ but also when L is an integral multiple of $2\pi/g$, because the curvature of the elastic curve of (39) contains spacial harmonics of the fundamental period L.

C. Circular Bend

Consider a circular bend of the lenslike medium as shown in Fig. 5. For this case, the curvature of the center axis of the medium is given as

$$\frac{1}{R(z)} = \frac{1}{R_c} = \text{constant}. \tag{44}$$

Substitute (44) into (14)–(22), and we have the field-distribution function $U(x,z)$. The result is represented by (26) in which $1/s^2(z)$, $\delta(z)$, $\delta'(z)$, g, and w are replaced, respectively, with $1/s_c^2(z)$, $\delta_c(z)$, $\delta_c'(z)$, $\tilde{g}$, and w_c given below.

$$\frac{1}{s_c^2(z)} = \frac{1}{s^2(0)} \left(\frac{\cos \tilde{g}z - j\,\dfrac{s^2(0)}{w_c^2} \sin \tilde{g}z}{\cos \tilde{g}z - j\,\dfrac{w_c^2}{s^2(0)} \sin \tilde{g}z} \right) \tag{45}$$

$$\delta_c(z) = \left(\delta(0) - \frac{1}{\tilde{g}^2 R_c}\right) \cos \tilde{g}z + \frac{\delta'(0)}{\tilde{g}} \sin \tilde{g}z + \frac{1}{\tilde{g}^2 R_c}, \tag{46}$$

$$\delta_c'(z) = \frac{d}{dz}\delta_c(z), \qquad \tilde{g} = g\left(1 - \frac{2}{g^2 R_c^2}\right)^{1/2},$$

$$w_c = \frac{1}{\sqrt{\tilde{g}k(0)}}. \tag{47}$$

Equation (46) represents the beam trajectory, and the spot size $w_c(z)$ and the propagation constant $\beta_\nu^{(c)}$ for the normal modes in the circular bend are obtained from (33) and (34), respectively, by replacing w and g in those expressions with the w_c and $\tilde{g}$ of (47).

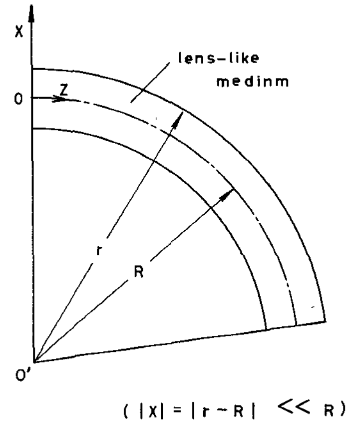


Fig. 5. Circular bend of the optical waveguide consisting of a lenslike medium.

If the input conditions of the light beam are chosen as

$$\delta(0) = \frac{1}{\tilde{g}^2 R_c}, \qquad \delta'(0) = 0, \qquad \frac{1}{s^2(0)} = \frac{1}{w_c^2} \tag{48}$$

the beam trajectory and the spot size are simplified, respectively, to

$$\delta_c(z) = \frac{1}{\tilde{g}^2 R_c} \qquad w_c(z) = w_c. \tag{49}$$

For the input conditions of (48), the light beam propagates along the bend without the fluctuations of the spot size and the beam trajectory, keeping the input spot size w_c (the characteristic spot size of the circular bend) and the input displacement of the beam center $1/(\tilde{g}^2 R_c)$. In this sense, (48) may be said as the matched input conditions for the circular bend.

If we assume $2/(g^2 R_c^2) \ll 1$ and neglect this term in the expressions (43)–(49), these expressions are reduced to the previous results obtained by the ray theory [5].

D. Linearly Tapered Bend

We consider a linearly tapered bend as shown in Fig. 6, in which the curvature of the center axis of the medium increases linearly with distance z from zero on the straight section to a maximum value $1/R_t$ as

$$\frac{1}{R(z)} = \frac{1}{R_t l} z, \qquad (0 \le z \le l) \tag{50}$$

where l is the length measured along the center axis of the medium $x = 0$.

Substituting (50) into (14)–(22), we obtain the field-distribution function $U(x,z)$. The main parameters for this case, which govern the response of the light beam, are given as follows:

$$\rho(z) = \left(1 - \frac{2z^2}{g^2 R_t^2 l^2}\right)^{1/2}$$

$$u(z) = \frac{-z}{g^3 R_t^2 l^2}\left(1 - \frac{2z^2}{g^2 R_t^2 l^2}\right)^{-3/2}$$

$$\theta(z) = \frac{1}{2}\left[z\left(1 - \frac{2z^2}{g^2 R_t^2 l^2}\right)^{1/2} + \frac{gR_t l}{\sqrt{2}} \operatorname{Sin}^{-1}\left(\frac{\sqrt{2}\,z}{gR_t l}\right)\right]$$

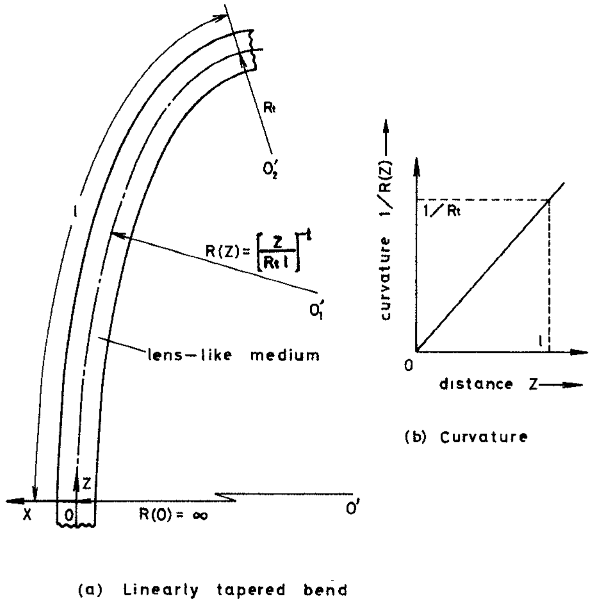


Fig. 6. Linearly tapered bend of the optical waveguide consisting of a lenslike medium.

$$w_c = w\left(= \frac{1}{\sqrt{gk(0)}}\right) \tag{51}$$

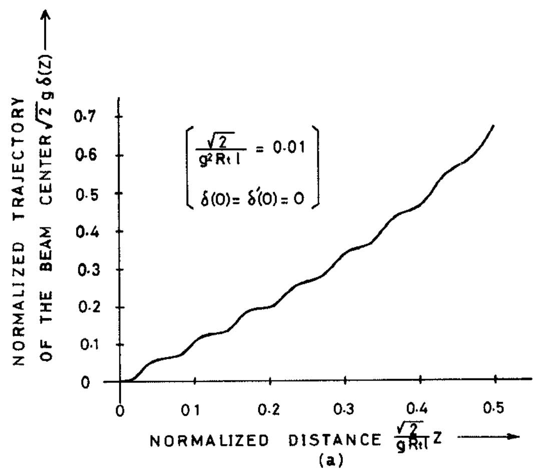
$$\begin{aligned}\delta(z) \cong & \left(1 - \frac{2z^2}{g^2R_t^2l^2}\right)^{-1/4} \\ & \cdot \left\{\delta(0)\cos g\theta(z) + \frac{\delta'(0)}{g}\sin g\theta(z)\right\} \\ & + \frac{1}{g^2R_tl}\left(1 - \frac{2z^2}{g^2R_t^2l^2}\right)^{-1/2} \\ & \cdot \left\{z\left(1 - \frac{z^2}{g^2R_t^2l^2}\right)^{-1/2} - \frac{1}{g}\sin g\theta(z)\right\}.\end{aligned} \tag{52}$$

Equation (52) gives the beam trajectory, and the spot size of a Gaussian beam $w(z)$ is calculated as

$$\begin{aligned}w(z) = & \left(1 - \frac{2z^2}{g^2R_t^2l^2}\right)^{-1/4} \mathrm{Re}^{-1/2}\left\{\frac{1}{s^2(0)}\right\} \\ & \cdot \left[\frac{1}{2}\left\{1 + \frac{w^4}{|s(0)|^4} + \left(1 - \frac{w^4}{|s(0)|^4}\right)\cdot\cos 2g\theta(z)\right\}\right. \\ & \left. + \mathrm{Im}\left\{\frac{w^2}{s^2(0)}\right\}\sin 2g\theta(z)\right]^{1/2}.\end{aligned} \tag{53}$$

Fig. 7(a) and (b) illustrates the calculated beam trajectory normalized by $(\sqrt{2}\,g)^{-1}$ in Fig. 7(a) and $\delta(0)$ in Fig. 7(b), as a function of the normalized distance $z/(gR_tl/\sqrt{2})$, assuming that $\sqrt{2}/(g^2R_tl) = 0.01$. In Fig. 7(a) the input conditions of the light beam are taken to be $\delta(0) = \delta'(0) = 0$, while in Fig. 7(b) $\delta(0) = 1/(g^3R_tl)$ and $\delta'(0) = 0$. Fig. 8 shows the calculated spot sizes normalized by the input value $w(0)$ as a function of the normalized distance $z/(gR_tl/\sqrt{2})$, in which the parameters are taken as $\sqrt{2}/(g^2R_tl) = 0.01$, and $w(0) = w/2$, w, and $2w$.

From these figures, it is seen that the light beam wanders away from the center axis of the medium as it propagates,



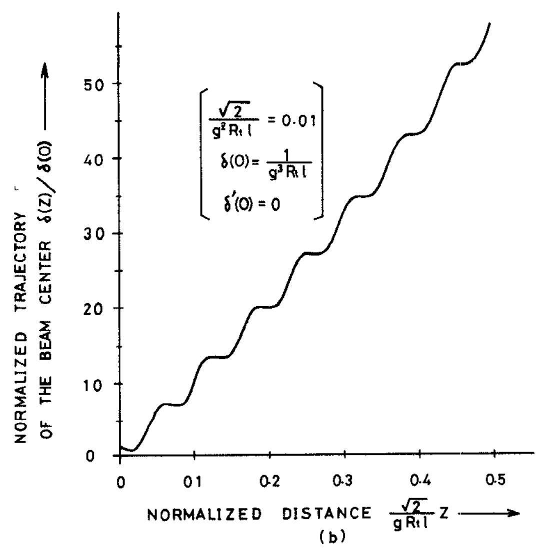


Fig. 7. Normalized beam trajectories of the light beam along the linearly tapered bend of the optical waveguide consisting of a lenslike medium. (a) For the case of $\delta(0) = \delta'(0) = 0$. (b) For the case of $\delta(0) \neq 0$ and $\delta'(0) = 0$.

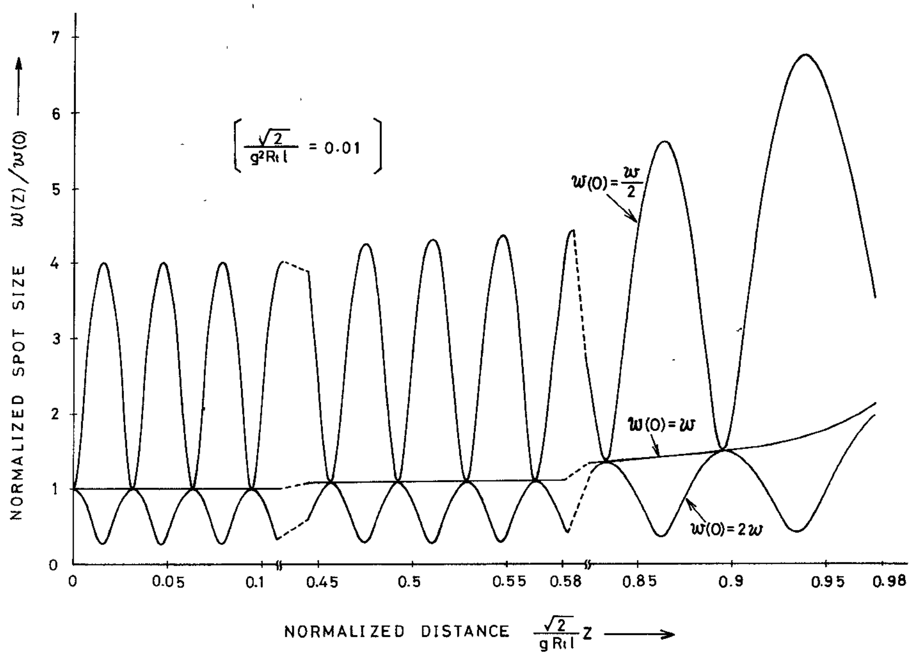


Fig. 8. Normalized spot sizes of a Gaussian beam along the linearly tapered bend of the optical waveguide consisting of a lenslike medium.

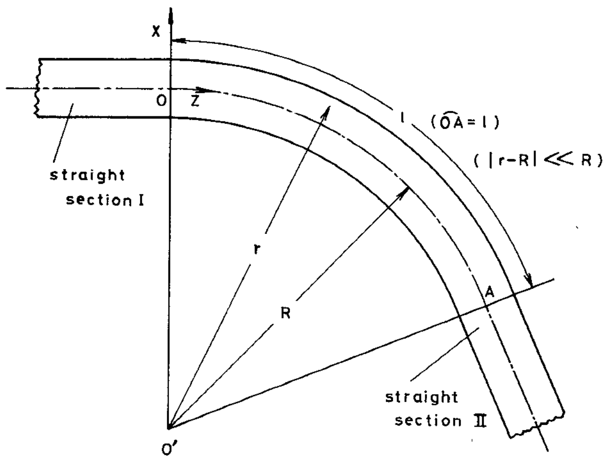


Fig. 9. Circular-bend section directly connected to the straight sections I and II without off-setting and tilting the center axis of the waveguide.

with slight undulations of the beam trajectory as well as fluctuations of the spot size, increasing the amplitudes and periods of the undulations and fluctuations. Especially, when the input wavefront coefficient $1/s^2(0)$ is taken as

$$\frac{1}{s^2(0)} = \frac{1}{w^2} \tag{54}$$

the spot size $w(z)$ becomes

$$w(z) = w\left(1 - \frac{2z^2}{g^2 R_t^{\,2} l^2}\right)^{-1/4} \tag{55}$$

and as a result its fluctuations are perfectly removed. Contrary to this, we cannot find any input conditions to remove the undulations of the beam trajectory. If we assume $2/(g^2R_t^{\,2}) \ll 1$, and hence that the terms with $1/(g^2R_t^{\,2})$ in expressions (51), (52) can be discarded, the beam trajectory of (52) completely agrees with the result of the previous analysis based on the ray theory [5].

IV. A Design Theory of the Circular Bend of Optical Waveguides Consisting of Lenslike Media

A. Effects of a Circular Bend

Suppose that a circular bend is inserted in an optical waveguide consisting of a lenslike medium, as shown in Fig. 9. The axis of this bend coincides with the axes of the straight sections and the permittivity is distributed in the straight sections I and II as

$$\varepsilon_s(x) = \varepsilon_s(0)(1 - g_s^{\,2}x^2) \tag{56}$$

and in the bend as

$$\varepsilon_c(r) = \varepsilon_c(R_c)\{1 - g_c^2(r - R_c)^2\} \tag{57}$$

where R_c is the radius of curvature satisfying $|r - R_c| \ll R_c$, and $\varepsilon_s(0)$ and $\varepsilon_c(R_c)$ represents the on-axis permittivities in the straight and circularly bent sections, respectively. g_s and g_c are the focusing parameters in the straight and bent sections, respectively.

Let us assume a Hermite–Gaussian beam as given by (23) at the entrance of the circular bend $z = 0$. Then the wavefront coefficient $1/s_2^2(z)$ and the beam trajectory $\delta_2(z)$ of the light beam in the outgoing straight section II are calculated from (45) and (46) as

$$\frac{1}{s_2^2(z)} = \frac{1}{s^2(0)}\left[\cos\tilde{g}_c l \cos g_s z - \frac{w_c^2}{w_s^2}\sin\tilde{g}_c l \sin g_s z - j\frac{s^2(0)}{w_s^2}\left(\cos\tilde{g}_c l \sin g_s z + \frac{w_s^2}{w_c^2}\sin\tilde{g}_c l \cos g_s z\right)\right] \cdot \left[\cos\tilde{g}_c l \cos g_s z - \frac{w_s^2}{w_c^2}\sin\tilde{g}_c l \sin g_s z - j\frac{w_s^2}{s^2(0)} \cdot \left(\cos\tilde{g}_c l \sin g_s z + \frac{w_c^2}{w_s^2}\sin\tilde{g}_c l \cos g_s z\right)\right]^{-1} \tag{58}$$

$$\delta_2(z) = \left(\delta(0) - \frac{1}{\tilde{g}_c^2 R_c}\right) \cdot \left[\cos\tilde{g}_c l \cos g_s z - \frac{\tilde{g}_c}{g_s}\sin\tilde{g}_c l \sin g_s z\right] + \frac{\delta'(0)}{g_s}\left[\cos\tilde{g}_c l \sin g_s z + \frac{g_s}{\tilde{g}_c}\sin\tilde{g}_c l \cos g_s z\right] + \frac{\cos g_s z}{\tilde{g}_c^2 R_c} \tag{59}$$

where

$$w_s = \frac{1}{\sqrt{g_s k_s(0)}}, \qquad w_c = \frac{1}{\sqrt{\tilde{g}_c k_c(R_c)}},$$

$$k_s(0) = \omega\sqrt{\mu\varepsilon_s(0)}, \qquad k_c(R_c) = \omega\sqrt{\mu\varepsilon_c(R_c)}, \tag{60}$$

$$\tilde{g}_c = g_c\sqrt{1 - \frac{2}{g_c^2 R_c^2}}. \tag{61}$$

From (58)–(61), we see that the effects of the circular bend are divided into a primary part due to a curvature $1/R_c$ and a secondary part due to $(1/R_c)^2$.[1]

B. Design Methods for Removing the Effects of a Circular Bend

In order to eliminate the effects of the circular bend clarified in the preceding section, we propose a new design method of the circular bend. In this method, a mode transducer is inserted between the straight section and the circularly bent section, by which beam modes are converted into normal modes in the circular bend.

Let us derive the design conditions of the mode transducer, according to the following examples.

1) Circular Bend of Lenslike Medium as a Mode Transducer: Let a circular bend be divided into three sections $\widehat{OA}$, $\widehat{AB}$, and $\widehat{BC}$, as in Fig. 10, wherein the radii of curvature are R_1, R_c, and R_1, and the lengths are l_1, l_c, and l_1, respectively. The permittivity in the sections OA and BC which play a role of mode transducer is given by

$$\varepsilon_1(r) = \varepsilon_1(R_1)\{1 - g_1^2(r - R_1)^2\} \tag{62}$$

and that in the section $\widehat{AB}$ is given by (57).

If the light beam is incident upon the bend off-axially and obliquely as given by (23), the beam trajectory in the outgoing straight section II is derived as

$$\begin{aligned}\delta_2(z) = {} & (K_1 \sin\tilde{g}_c l_c + K_2\cos\tilde{g}_c l_c + K_3) \cdot \cos g_s(z - 2l_1 - l_c) \\ & + \frac{1}{g_s}(K_1'\sin\tilde{g}_c l_c + K_2'\cos\tilde{g}_c l_c + K_3') \cdot \sin g_s(z - 2l_1 - l_c) \\ & + \delta(0)\left[\left\{\cos 2\tilde{g}_1 l_1 \cos\tilde{g}_c l_c - \frac{1}{2}\left(\frac{\tilde{g}_1}{\tilde{g}_c} + \frac{\tilde{g}_c}{\tilde{g}_1}\right) \cdot \sin 2\tilde{g}_1 l_1 \sin\tilde{g}_c l_c\right\} \cdot \cos g_s(z - 2l_1 - l_c)\right. \\ & - \frac{\tilde{g}_1}{g_s}\left\{\sin 2\tilde{g}_1 l_1 \cos\tilde{g}_c l_c + \left(\frac{\tilde{g}_c}{\tilde{g}_1}\cos^2\tilde{g}_1 l_1 - \frac{\tilde{g}_1}{\tilde{g}_c}\sin^2\tilde{g}_1 l_1\right) \cdot \sin\tilde{g}_c l_c\right\} \cdot \left.\sin g_s(z - 2l_1 - l_c)\right] \\ & + \frac{\delta'(0)}{g_s}\left[\frac{g_s}{\tilde{g}_1}\left\{\sin 2\tilde{g}_1 l_1 \cos\tilde{g}_c l_c + \left(\frac{\tilde{g}_1}{\tilde{g}_c}\cos^2\tilde{g}_1 l_1 - \frac{\tilde{g}_c}{\tilde{g}_1}\sin^2\tilde{g}_1 l_1\right) \cdot \sin\tilde{g}_c l_c\right\} \cdot \cos g_s(z - 2l_1 - l_c)\right. \\ & + \left\{\cos 2\tilde{g}_1 l_1 \cos\tilde{g}_c l_c - \frac{1}{2}\left(\frac{\tilde{g}_1}{\tilde{g}_c} + \frac{\tilde{g}_c}{\tilde{g}_1}\right) \cdot \sin 2\tilde{g}_1 l_1 \sin\tilde{g}_c l_c\right\} \cdot \left.\sin g_s(z - 2l_1 - l_c)\right]\end{aligned} \tag{63}$$

where

$$K_1 = \frac{\tilde{g}_c}{\tilde{g}_1}\left(\frac{1}{\tilde{g}_c^2 R_c} - \frac{1}{\tilde{g}_1^2 R_1}\right)\sin\tilde{g}_1 l_1 + \frac{1}{\tilde{g}_1^2 R_1}\left(\frac{\tilde{g}_c}{\tilde{g}_1} + \frac{\tilde{g}_1}{\tilde{g}_c}\right)\cos\tilde{g}_1 l_1 \sin\tilde{g}_1 l_1$$

[1] Equations (58)–(61) further include the effects of a bend due to higher order terms in $(1/R_c)^3, (1/R_c)^4, \cdots$. However, under a Hermite–Gaussian approximation and within the accuracy of the phase constant adopted in this paper, it would be meaningless to take into consideration corrections higher than the third order.

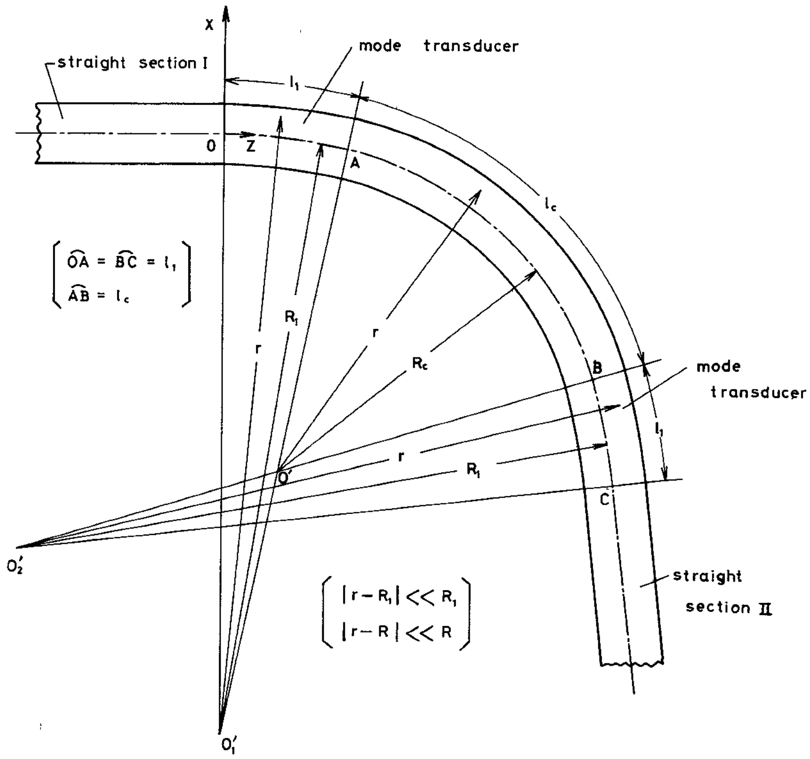


Fig. 10. Proposed design method of the circular bend for eliminating the effects of the bend.

$$K_2 = -\left\{\left(\frac{1}{\tilde{g}_c{}^2 R_c} - \frac{1}{\tilde{g}_1{}^2 R_1}\right)\cos \tilde{g}_1 l_1 + \frac{1}{\tilde{g}_1{}^2 R_1}(\cos^2 \tilde{g}_1 l_1 - \sin^2 \tilde{g}_1 l_1)\right\}$$

$$K_3 = \left(\frac{1}{\tilde{g}_c{}^2 R_c} - \frac{1}{\tilde{g}_1{}^2 R_1}\right)\cos \tilde{g}_1 l_1 + \frac{1}{\tilde{g}_1{}^2 R_1} \tag{64}$$

$$K_1' = \tilde{g}_c\left(\frac{1}{\tilde{g}_c{}^2 R_c} - \frac{1}{\tilde{g}_1{}^2 R_1}\right)\cos \tilde{g}_1 l_1 + \frac{1}{\tilde{g}_1 R_1}\left(\frac{\tilde{g}_c}{\tilde{g}_1}\cos^2 \tilde{g}_1 l_1 - \frac{\tilde{g}_1}{\tilde{g}_c}\sin^2 \tilde{g}_1 l_1\right)$$

$$K_2' = \left\{\frac{\tilde{g}_1}{\tilde{g}_c{}^2 R_c} - \frac{1}{\tilde{g}_1 R_1}(1 - 2\cos \tilde{g}_1 l_1)\right\} \cdot \sin \tilde{g}_1 l_1$$

$$K_3' = \left(\frac{1}{\tilde{g}_1 R_1} - \frac{\tilde{g}_1}{\tilde{g}_c{}^2 R_c}\right)\sin \tilde{g}_1 l_1 \tag{65}$$

$$\tilde{g}_1 = g_1\sqrt{1 - \frac{2}{g_1{}^2 R_1{}^2}} \tag{66}$$

and $\tilde{g}_c$ is given by (61).

The first and second terms in (63) including K_1, K_2, K_3 and K_1', K_2', K_3' given by (64) and (65) are related to the curvatures $1/R_1$ and $1/R_c$, the first-order effect in the bend. The third and fourth terms in (63) are, as is clear from (61) and (66), related to the curvatures squared $(1/R_1)^2$ and $(1/R_c)^2$, the second-order effect in the bend. For $R_1 \to \infty$, $R_c \to \infty$, and $g_1 = g_c = g_s$, (63) simplifies to

$$\delta_2(z) = \delta(0)\cos g_s z + \frac{\delta'(0)}{g_s}\sin g_s z. \tag{67}$$

Equation (67) would be obtained if $\widehat{OA}$, $\widehat{AB}$, and $\widehat{BC}$ were all straight sections.

Equating (63) to (67) for any l_c, $\delta(0)$, and $\delta'(0)$, we have

$$K_1 = K_2 = K_3 = 0$$
$$K_1' = K_2' = K_3' = 0 \tag{68}$$

$$\frac{1}{2}\left(\frac{\tilde{g}_1}{\tilde{g}_c} + \frac{\tilde{g}_c}{\tilde{g}_1}\right) = 1, \quad \frac{g_s}{\tilde{g}_1} = 1. \tag{69}$$

Equations (68) and (69) represent the conditions for removing the first-order and second-order effects of the bend, respectively.

In order to remove the effect of the bend completely up to the second order included in the spot size, we must further require the matching conditions for the spot sizes in the straight and circularly bent sections as

$$\frac{1}{\sqrt{g_s k_s(0)}} = \frac{1}{\sqrt{\tilde{g}_c k_c(R_c)}} = \frac{1}{\sqrt{\tilde{g}_1 k_1(R_1)}} \tag{70}$$

with

$$k_1(R_1) = \omega\sqrt{\mu\varepsilon_1(R_1)}. \tag{71}$$

From (64)–(66) and (68)–(71), we can determine l_1, R_1, g_1, and g_c. As a result, we obtain a design method as

$$l_1 = \frac{(2N+1)\pi}{g_s}, \qquad (N = 0,1,2,\cdots)$$
$$R_1 = 2R_c \qquad \varepsilon_1(R_1) = \varepsilon_c(R_c) = \varepsilon_s(0)$$
$$g_1 = g_s\left(1 + \frac{1}{2g_s^2R_c^2}\right)^{1/2} \qquad g_c = g_s\left(1 + \frac{1}{g_s^2R_c^2}\right)^{1/2} \tag{72}$$

where we have assumed that l_c is arbitrary and $R_c \neq \infty$.

The electromagnetic fields of the outgoing straight section II derived as a result of the design method of (72) agree with those which would be obtained by assuming that $\widehat{OA}$, $\widehat{AB}$, and $\widehat{BC}$ are all straight ($R_1 = R_c = \infty$). This means that the effect of the bend has been completely removed. Moreover, the fields in the circularly bent section $\widehat{AB}$ coincide with those in the straight section II, if the beam trajectory $\delta_2(z)$ is replaced with $\delta_c(z)$ as given by

$$\delta_c(z) = \frac{1}{g_s^2R_c} + \delta(0)\cos g_sz + \frac{\delta'(0)}{g_s}\sin g_sz. \tag{73}$$

The results are identical with the fields which would be obtained by displacing the axis of the circular bend towards the center of the curvature by $1/(g_s^2R_c)$ [5], [9].

If the radius of curvature R_c is large enough to satisfy the condition

$$\frac{2}{g_s^2R_c^2} \ll 1 \tag{74}$$

the design method of (72) is simplified to method (a) of Table I. In this case, the method requires that the length l_1 must be equal to an odd multiple of π/g_s and that the radius of curvature R_1 be equal to $2R_c$, but the focusing parameters g_1 and g_c may not vary from those of the straight section.

2) *Circular Bend of Tapered Lenslike Medium as a Mode Transducer:* Let us replace sections $\widehat{OA}$ and $\widehat{BC}$ of Fig. 10 by tapered lenslike media with a raised-cosine taper [12]. We assume that the permittivity is given in $\widehat{OA}$ as

$$\varepsilon_{OA}(r,z) = \varepsilon_1(R_1)\left[1 - g_{0t}^2\left(\frac{1 + a\cos\frac{\pi z}{l_1}}{1+a}\right)^2(r - R_1)^2\right] \tag{75}$$

and in $\widehat{BC}$ as

$$\varepsilon_{BC}(r,z) = \varepsilon_1(R_1)\left[1 - g_{0t}^2\left\{\frac{1 - a\cos\frac{\pi(z - l_1 - l_c)}{l_1}}{1+a}\right\}^2 \cdot (r - R_1)^2\right] \tag{76}$$

and also that in $\widehat{AB}$ it is given by (57). In the expressions (75) and (76), g_{0t} and a are independent of r and z ,and we assume $0 < |a| < 1$.

TABLE I
DESIGN CONDITIONS FOR ELIMINATING THE EFFECTS OF THE CIRCULAR BEND IN WHICH $2/(g_s^2R_c^2) \ll 1$ IS ASSUMED

	Simplified Design Conditions
(a)	$l_1 = \frac{(2N+1)\pi}{g_s}$ (N = 0, 1, 2 -----) l_c = arbitrary , R_c = arbitrary $\neq \infty$ $g_1 = g_c = g_s$ $\varepsilon_1(R_1) = \varepsilon_c(R_c) = \varepsilon_s(0)$ $R_1 = 2R_c$
(b)	$l_1 = \frac{(2N+1)\pi}{g_s}$ (N = 0, 1, 2 -----) l_c = arbitrary , R_c = arbitrary $\neq \infty$ $g_{0t} = g_s$, $a = 1/(2g_s^2R_c^2)$ $\varepsilon_1(R_1) = \varepsilon_c(R_c) = \varepsilon_s(0)$ $R_1 = R_c\left[1 + (\frac{g_c}{g_s})^{3/2}\right]$
(c)	$l_1 = \frac{2N\pi}{g_s}$ (N = 1, 2, 3 -----) l_c = arbitrary , R_c = arbitrary $\neq \infty$ $g_s = g_c$, $\varepsilon_c(R_c) = \varepsilon_s(0)$

(a) The circular bend of a lenslike medium as a mode transducer.
(b) The circular bend of a tapered lenslike medium as a mode transducer.
(c) The linearly tapered bend of a lenslike medium as a mode transducer.

Suppose that an Hermite–Gaussian beam is incident off-axially and obliquely and determine l_1, a, g_{0t}, and R_1 by the same procedure as was done in the preceding subsection. Then we have the following design method:

$$l_1 = \frac{(2N+1)\pi}{\tilde{g}_{0t}}, \qquad (N = 0,1,2,\cdots)$$
$$a = \frac{1 - \sigma^2 - g_s^2g_{0t}^{-2}(1 - \sigma^2 - 2g_s^{-2}R_c^{-2})}{[1 + \{\sigma^2 + g_s^2g_{0t}^{-2}(1 - \sigma^2 - 2g_s^{-2}R_c^{-2})\}^{1/2}]^2}$$
$$g_{0t} = g_s(1 + 2g_s^{-2}R_1^{-2})^{1/2}$$
$$R_1 = R_c\{1 + \sigma^{3/2}(1 - 2g_c^{-2}R_c^{-2})^{3/4}\} \tag{77}$$

with

$$\tilde{g}_{0t} = \frac{g_{0t}}{2}\{(1 - 2g_{0t}^{-2}R_c^{-2})^{1/2} + 1\}, \qquad \sigma = g_c/g_s \tag{78}$$

where we have assumed that $\varepsilon_1(R_1) = \varepsilon_c(R_c) = \varepsilon_s(0)$, $R_1 > R_c$, and l_c is arbitrary.

If $g_c = g_s$, this method agrees with that proposed in [12] to remove the undulations of the beam trajectory and the spot size for a light beam incident onto the axis with the input conditions $\delta(0) = \delta'(0) = 0$, $s(0) = w_s$. Here we must note that by this design method, unlike the method of (72), a lenslike medium having the same focusing parameter as in the straight section can be applied to the section $\widehat{AB}$, even if the condition of (74) is not satisfied.

The fields in the straight section II derived as a result of this method agree with those which would be obtained if sections $\widehat{OA}$, $\widehat{AB}$, and $\widehat{BC}$ were made straight with a length

$$\Delta_c = 2l_1 + l_c\left(1 - \frac{g_c}{g_s}\sqrt{1 - \frac{2}{g_c^2 R_c^2}}\right). \quad (79)$$

Thus we see that the effect of the bend has been removed perfectly except for the phase shift due to the term $(l_c g_c/g_s)\sqrt{1 - 2/(g_c^2 R_c^2)}$.

If the condition of (74) is satisfied, the design method of (77) is reduced to method (b) of Table I.

3) Linearly Tapered Bend of Lenslike Medium as a Mode Transducer: Let sections $\widehat{OA}$ and $\widehat{BC}$ of Fig. 10 be replaced by linearly tapered bends of the lenslike medium as investigated in the preceding section. We assume the curvature of the tapered bend in $\widehat{OA}$ as

$$\frac{1}{R_{OA}(z)} = \frac{z}{R_c l_1}, \qquad (0 \le z \le l_1) \quad (80)$$

and in $\widehat{BC}$ as

$$\frac{1}{R_{BC}(z)} = -\frac{z - l_c - 2l_1}{R_c l_1}, \qquad (l_c + l_1 \le z \le l_c + 2l_1). \quad (81)$$

For simplicity, let us assume that (74) is satisfied and also that $g_s = g_c$ and $s(0) = w$. Then, as is clear from (51)–(53), the effects of the linearly tapered bends appear only in the undulations of the beam trajectory. In the following, therefore, we restrict our attention to the beam trajectory.

If the light beam as given by (23) is incident on the tapered bend section $\widehat{OA}$ and propagates through the circular bend $\widehat{AB}$ and the tapered bend $\widehat{BC}$, the beam trajectory $\delta_2(z)$ in the outgoing straight section II is derived as

$$\begin{aligned}\delta_2(z) = {} & \delta(0)\cos g_s z + \frac{\delta'(0)}{g_s}\sin g_s z \\ & + \frac{1}{g_s^3 R_c l_1}\left[(\sin g_s l_1 - \sin 2g_s l_1)\cos g_s l_c\right. \\ & + (\cos g_s l_1 - \cos 2g_s l_1)\sin g_s l_c + \left.\sin g_s l_1\right] \\ & \cdot \cos g_s(z - l_c - 2l_1) \\ & + \frac{1}{g_s^3 R_c l_1}\left[\cos g_s l_1 - 1\right. \\ & + (\cos g_s l_1 - \cos 2g_s l_1)\cos g_s l_c \\ & - \left.(\sin g_s l_1 - \sin 2g_s l_1)\sin g_s l_c\right] \\ & \cdot \sin g_s(z - l_c - 2l_1). \end{aligned} \quad (82)$$

If we put

$$l_1 = \frac{2\pi N}{g_s}, \qquad N = 1,2,3,\cdots \quad (83)$$

equation (82) simplifies to the same form as (67), in which the effects of the circular bend have been perfectly eliminated. The design condition of this method is listed in method (c) of Table I.

Numerical results are given in Tables II–IV, which were obtained by applying the methods of (72), (77), and (c) of Table I.

V. Analogies of Optical Waveguides Consisting of Lenslike Media to Circular TE_{01} Waveguides

A. Sinusoidal Bend

As stated before, in the sinusoidal bend, the divergence phenomenon of the beam trajectory occurs when the bending period p is just equal to $2\pi/g$. Let us compare this phenomenon with the mode conversion at the sinusoidal bend of the TE_{01} circular waveguide. As is well known, in the case of the TE_{01} circular waveguide, when the bending period p is equal to the beat wavelength between the TE_{01} mode and any coupled mode, then continuous power conversion occurs from the TE_{01} mode to that coupled mode, and results in a large mode-conversion loss [14], [15]. On the other hand, if we derive the beat wavelength λ_b of the optical waveguide consisting of a lenslike medium from (34) according to the definition $1/\lambda_b = 1/\lambda_\nu - 1/\lambda_{\nu-1}$, we find that the quantity $2\pi/g$ previously mentioned is nothing but the beat wavelength λ_b between the normal modes of the optical waveguide. Therefore, the divergence phenomenon of the beam trajectory in the lenslike medium may be considered as resulting from mode conversion to higher order modes, which occurs when the bending period p is equal to the beat wavelength $2\pi/g$.

B. Serpentine Bend

As clarified before, in the serpentine bend of the optical waveguide, the divergence phenomenon of the beam trajectory arises when the support interval L is equal to an integral multiple of $2\pi/g$ ($L = 2\pi n/g$: $n = 1,2,3,\cdots$). As in the preceding subsection, we compare this phenomenon with the mode conversion at the serpentine bend of the TE_{01} circular waveguide. It is known [14] that in the case of the TE_{01} circular waveguide, if the support interval L is equal to an integral multiple of the beat wavelength between the TE_{01} mode and any coupled mode, the continuous power conversion occurs from the TE_{01} mode to that coupled mode, resulting in a large mode-conversion loss. Therefore, we may interpret the divergence phenomenon of the beam trajectory as resulting from mode conversion to higher order modes, which occurs when the support interval L is equal to an integral multiple of the beat wavelength $2\pi/g$, as previously stated.

C. Circular Bend

In the preceding section, we have proposed a new design method of the circularly bent section of the optical waveguide, consisting of a lenslike medium, to remove the effects of the bend. Several other methods have also been proposed in the past [5], [9], [12], [20]. For example, Unger [5] and Suematsu and Fukinuki [9] proposed to displace the center axis at the bend towards the center of curvature by a small amount δ, as shown in Fig. A of Table V: Unger [5] also

TABLE II
NUMERICAL EXAMPLES FOR THE DESIGN METHOD UTILIZING CIRCULAR BENDS OF LENSLIKE MEDIA AS MODE TRANSDUCERS

Type of lens-like medium	Straight section		Circularly bent section $\widehat{AB}$		Circulary bent sections $\widehat{OA}$ and $\widehat{BC}$ (Mode transducer sections)		
	on-axis permittivity $\varepsilon_s^*(0)$	focusing parameter g_s	radius of curvature R_c	focusing parameter g_c	radius of curvature R_1	focusing parameter g_1	length of mode transducer l_1
SELFOC	2·5	3 (mm)$^{-1}$	10 mm	3·003 (mm)$^{-1}$	20 mm	3·001 (mm)$^{-1}$	1·047 mm
		0·3 (mm)$^{-1}$	10 cm	0·3003 (mm)$^{-1}$	20 cm	0·3001 (mm)$^{-1}$	10·47 mm
Gas-lens	1·0	0·447 m^{-1}	3 Km	0·4470 m^{-1}	6 Km	0·4470 m^{-1}	7·028 m

TABLE III
NUMERICAL EXAMPLES FOR THE DESIGN METHOD UTILIZING CIRCULAR BENDS OF TAPERED LENSLIKE MEDIA AS MODE TRANSDUCERS

Type of lens-like medium	Straight section		Circularly bent section $\widehat{AB}$		Circular bent sections $\widehat{OA}$ and $\widehat{BC}$ (Mode transducer sections)			
	on-axis permittivity $\varepsilon_s^*(0)$	focusing parameter g_s	radius of curvature R_c	focusing parameter g_c	radius of curvature R_1	length of mode transducer l_1	constants of taper	
							a	g_{0t}
SELFOC	2 5	3 (mm)$^{-1}$	1 0 mm	3·3 (mm)$^{-1}$	21·52 mm	1·031 mm	-1.050×10^{-3}	3·046 (mm)$^{-1}$
		3 (mm)$^{-1}$	1 0 mm	3·03 (mm)$^{-1}$	20 13 mm	1 050 mm	5.278×10^{-5}	3 001 (mm)$^{-1}$
		0·3 (mm)$^{-1}$	1 0 cm	0 3 (mm)$^{-1}$	19 98 cm	1 050 cm	$5\ 552\times10^{-4}$	0·3001 (mm)$^{-1}$
Gas-lens	1 0	0·477 m^{-1}	3 km	0·477 m^{-1}	6 km	7·028 m	$2\ 780\times10^{-7}$	0 4470 m^{-1}

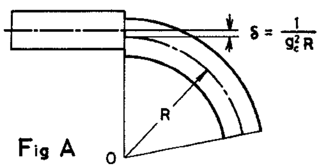
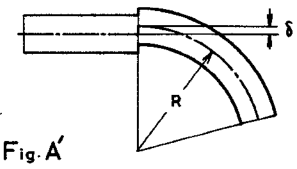
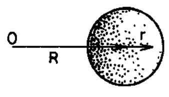
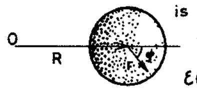
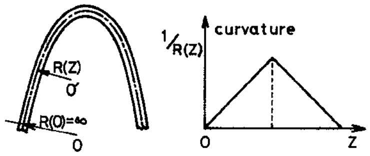
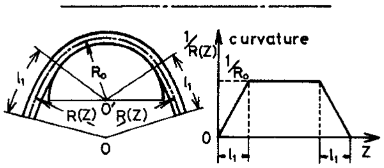
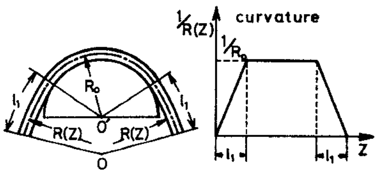
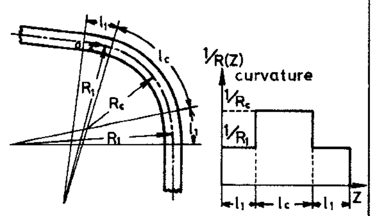
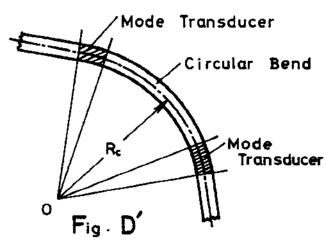
TABLE IV
NUMERICAL EXAMPLES FOR THE DESIGN METHOD UTILIZING LINEARLY TAPERED BENDS OF LENSLIKE MEDIA AS MODE TRANSDUCERS

Type of lens-like medium	Straight section		Circularly bent section $\widehat{AB}$		Circularly bent sections $\widehat{OA}$ and $\widehat{BC}$ (Mode transducer sections)	
	on-axis permittivity $\varepsilon_s^*(0)$	focusing parameter g_s	radius of curvature R_c	focusing parameter g_c	focusing parameter g	length of mode transducer l_1
SELFOC	2 5	3 (mm)$^{-1}$	1 0 mm	3 (mm)$^{-1}$	3 (mm)$^{-1}$	2·094 mm
		0·3 (mm)$^{-1}$	1 0 cm	0·3 (mm)$^{-1}$	0·3 (mm)$^{-1}$	20·94 mm
Gas-lens	1 0	0·447 m^{-1}	3 km	0 447 m	0·447 m^{-1}	14 056 m

proposed to taper the radius of curvature R without displacing the center axis, as shown in Fig. C_1 of that table. These methods were devised to eliminate the undulation of the beam trajectory without considering the fluctuations of the spot size. On the other hand, Imai and Matsumoto [20] proposed not only to displace the center axis at the bend by δ, as shown in Fig. A of Table V, in order to eliminate the undulation of the beam trajectory, but also properly to alter the distribution of the permittivity at the bend, as shown in Fig. B of that table, in order to eliminate the fluctuation of the spot size.

Table V compares the aforementioned design methods, including the authors' methods proposed in the preceding section, with several conventional design methods for a circular bend of the TE_{01} circular waveguide [13]–[19]. It is interesting to note the following correspondences. The method of Unger [5] or Suematsu and Fukinuki [9] for eliminating the undulations of the beam trajectory corresponds to the method proposed by Kumagai and Yoshida [17], [18] to eliminate coupling from the TE_{01} to the TE_{1n} modes, as shown in Fig. A′ of Table V. The method of Imai and Matsumoto [20] or Suematsu and Fukinuki [9] for eliminating the fluctuations of the spot size (Fig. B) corresponds to the method proposed by Morgan [16] to eliminate coupling from the TE_{01} to the TM_{11} modes as shown in Fig. B′. On the other hand, the method of the

TABLE V
DESIGN METHODS OF CIRCULAR BENDS OF THE OPTICAL WAVEGUIDE CONSISTING OF A LENSLIKE MEDIUM AND THE CIRCULAR TE_{01} WAVEGUIDE

	OPTICAL WAVEGUIDE CONSISTING OF A LENS-LIKE MEDIUM	CIRCULAR TE_{01} WAVEGUIDE
Center axis is displaced by δ	 Fig A Fluctuation of the beam trajectory can be eliminated [5], [9].	 Fig. A′ Coupling to TE_{1n} modes can be eliminated [17], [18].
Permittivity distribution is varied	 On-axis permittivity ε(R) or focusing parameter g is varied [9], [20]. $\varepsilon(r) = \varepsilon(R)\left[1-g^2(r-R)^2\right]$ Fig. B Fluctuation of spot size can be eliminated.	 Permittivity distribution is varied according to [16], [17] as $\varepsilon(r,\varphi) = \varepsilon_0\left[1-\frac{2r}{R}\cos\varphi\right]$ Fig. B′ Coupling to TM_{11} modes can be eliminated.
Radius of curvature is tapered	 Fig C_1  Fig. C_2 Fluctuation of the beam trajectory can be eliminated [5], [this paper].	 Fig. C′ Coupling to TE_{1n} modes can be eliminated [15].
Mode is converted to normal mode of bend section	 Fig. D All fluctuations can be eliminated [12], [this paper].	 Fig. D′ Coupling to all of unwanted modes can be eliminated [19].

same authors [9], [20] for eliminating both the undulation of the beam trajectory and the fluctuation of the spot size by the combined use of the schemes of Figs. A and B corresponds to the method proposed by Kumagai and Yoshida [17], [18] to eliminate both the coupling between TE_{01} and TE_{1n} and that between TE_{01} and TM_{11} by the combined use of the schemes of Figs. A′ and B′. Also, the method of Unger [5] as shown in Fig. C_1 or that proposed in this paper as shown in Fig. C_2 for eliminating the undulation of the beam trajectory corresponds to the method proposed by Unger [15], as shown in Fig. C′, to eliminate coupling between TE_{01} and TE_{1n} modes. Finally, the methods of (72) and (77) proposed in this paper for the simultaneous elimination of the undulations of the beam trajectory and the spot size (Fig. D) correspond to the method of Miller [19] (Fig. D′): the input beam in the former or the input TE_{01} mode in the latter is converted by way of the section $\widehat{OA}$ into the normal modes of the circularly bent section $\widehat{AB}$.

From these results we find that the methods for eliminating the undulations of the beam trajectory in the optical waveguide with a square-law lenslike medium correspond to those for preventing the mode conversion from TE_{01} to TE_{1n} modes at a circular bend of the circular TE_{01} waveguide, while the methods for eliminating the fluctuations of the spot size in the former correspond to those for preventing the mode conversion from TE_{01} to TM_{11} modes in the latter.

VI. Conclusion

Propagation behavior of light beams along curved lens-like media has been analyzed with the help of the convenient method of analysis based on the approximate wave theory. The sinusoidal and serpentine bends as well as the circular bend and linearly tapered bend of the optical waveguides with a square-law lenslike medium have been investigated in detail theoretically and numerically, and the results have been compared with the previous results by the ray theory. A new design method of the circular bend of the optical waveguide has been proposed, by which the effects of the circular bend can be completely removed without offsetting and tilting the center axis of the bend, unlike the previous methods, and the numerical examples have been presented. Further, we have shown the analogies between the optical waveguide with a square-law lenslike medium and the circular TE_{01} waveguide for the cases of sinusoidal bends, serpentine bends, and circular bends. These analogies are attributable to the multimode characteristic of any transmission system. Therefore, the analogies shown in this paper are not restricted to the particular waveguides studied in the present paper, but they exist, in general, among various multimode transmission systems. The analogies may be utilized to explain the characteristics of one system or to design one system from the known facts about the other system. This may necessitate treating the transmission system on the basis of the wave theory or to study the behavior of the electromagnetic wave in terms of wave modes. From this point of view, we have treated the optical waveguide consisting of a square-law lenslike medium as a multimode transmission system, and have discussed the characteristics of this waveguide in comparison with those of the circular TE_{01} waveguide, one of the most thoroughly studied multimode transmission systems.

Acknowledgment

The authors wish to thank the reviewers for their useful comments that helped to derive the complete form of the wave equation (5), including the additional small terms consisting of the radius of curvature $R(z)$ and its first and second derivatives with respect to z.

References

[1] S. E. Miller, "Integrated optics: An introduction," *Bell Syst. Tech. J.*, vol. 48, pp. 2059–2069, Sept. 1969.
[2] T. Uchida *et al.*, "Optical characteristics of a light-focusing fiber guide and its applications," *IEEE J. Quantum Electron.*, vol. QE-6, pp. 606–612, Oct. 1970.
[3] D. W. Berrman, "A lens or light guide using convectively distorted thermal gradients in gases," *Bell Syst. Tech. J.*, vol. 43, pp. 1469–1475, July 1964.
[4] P. K. Tien, J. P. Gordon, and J. R. Whinnery, "Focusing of a light beam of Gaussian field distribution in continuous and periodic lenslike media," *Proc. IEEE*, vol. 53, pp. 129–136, Feb. 1965.
[5] H. G. Unger, "Light beam propagation in curved schlieren guides," *Arch. elekt. Übertragung*, vol. 19, pp. 189–198, April 1965.
[6] S. E. Miller, "Light propagation in generalized lens-like media," *Bell Syst. Tech. J.*, vol. 44, pp. 2017–2064, Nov. 1965.
[7] E. A. Marcatili, "Mode in a sequence of thick astigmatic lens-like focusers," *Bell Syst. Tech. J.*, vol. 43, pp. 2887–2904, Nov. 1964.
[8] H. Kogelnik, "On the propagation of Gaussian beams of light through lens-like media including those with a loss or gain variation," *Appl. Opt.*, vol. 4, pp. 1562–1569, Dec. 1965.
[9] Y. Suematsu and H. Fukinuki, "Analysis of idealized light waveguide using gas lens," *Jour. Inst. Elec. Commun. Engrs. Jap.*, vol. 48, pp. 1684–1969, Oct. 1965.
[10] S. Sawa and N. Kumagai, "A new method of analysis of the waveguide consisting of lens-like medium," *Trans. Inst. Elec. Commun. Engrs. Jap.*, vol. 52-B, pp. 624–631, Oct. 1969.
[11] S. Sawa, "Propagation of light beam through lens-like media with complex permittivity," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 566–575, July 1975.
[12] ——, "A theoretical study of light beams guided along tapered lens-like media, and their applications," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 77–93, Feb. 1976.
[13] W. J. Albersheim, "Propagation of TE_{01} waves in curved waveguides," *Bell Syst. Tech. J.*, vol. 28, pp. 1–32, Jan. 1949.
[14] H. G. Unger, "Circular electric wave transmission through serpentine bends," *Bell Syst. Tech. J,*. vol. 36, pp. 1279–1291, Sept. 1957.
[15] ——, "Normal mode bends for circular electric waves," *Bell Syst. Tech. J.*, vol. 36, pp. 1292–1307, Sept. 1957.
[16] S. P. Morgan, "Theory of curved circular waveguide containing an inhomogeneous dielectric," *Bell Syst. Tech. J.*, vol. 36, pp. 1209–1251, Sept. 1957.
[17] N. Kumagai and K. Yoshida, "A study on the circular bends of circular TE_{01} waveguide," *Papers of Technical Group on Microwaves, Inst. Elect. Commun. Engrs. Jap.*, Feb. 1966.
[18] ——, "A study on the bends of circular TE_{01} waveguide," *Natl. Conv. Record of Inst. Elect. Commun. Engrs. Jap.*, 389, 1966.
[19] S. E. Miller, "Notes on methods of transmitting the circular electric wave around bends," *Proc. IRE*, vol. 40, pp. 1104–1113, Sept. 1952.
[20] M. Imai and T. Matsumoto, "Dielectric-constant distribution for suppressing mode conversion due to bends in gas lenses," *Trans. Inst. Elect. Commun. Engrs. Jap.*, vol. 51-B, pp. 598–599, Dec. 1958.
[21] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics.* New York: McGraw-Hill, 1953, ch. 9.